2011-03-08

Mean Square Estimate for Primitive Lattice Points in Convex Planar Domains

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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April 2011

Abstract

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The Gauss circle problem in classical number theory concerns the estimation of $N(x) = \{ (m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 \leq x \}$, the number of integer lattice points inside a circle of radius $\sqrt{x}$. Gauss showed that $P(x) = N(x) - \pi x$ satisfies $P(x) = O(\sqrt{x})$. Later Hardy and Landau independently proved that $P(x) = \Omega_{-}(x^{1/4}(\log x)^{1/4})$. It is conjectured that $\inf \{ \theta \in \mathbb{R} : P(x) = O(x^{\theta}) \} = \frac{1}{4}$. I. Kátai [10] showed that $\int_0^X |P(x)|^2 \ dx = \beta X^{3/2} + O(X(\log X)^2)$.

Similar results to those of the circle have been obtained for regions $\mathcal{D} \subset \mathbb{R}^2$ which contain the origin and whose boundary $\partial\mathcal{D}$ satisfies sufficient smoothness conditions. Denote by $P_{\mathcal{D}}(x)$ the similar error term to $P(x)$ only for the domain $\mathcal{D}$. W. G. Nowak showed that, under appropriate conditions on $\partial\mathcal{D}$, $P_{\mathcal{D}}(x) = \Omega_{-}(x^{1/4}(\log x)^{1/4})$ ([12]) and that $\int_0^X |P_{\mathcal{D}}(x)|^2 \ dx = O(X^{3/2})$ ([13]).

A result similar to Nowak's mean square estimate is given in the case where only "primitive" lattice points, $\{ (m_1, m_2) \in \mathbb{Z}^2 : \gcd(m_1, m_2) = 1 \}$, are counted in a region $\mathcal{D}$, on assumption of the Riemann Hypothesis.

Keywords: Number Theory, Lattice Points, Hlawka Zeta Function, Mean Square Estimate, Gauss Circle Problem, Riemann Hypothesis

Acknowledgments

This work would not be possible without the patience of several important people. At the top of the list are my advisor who was patient during the entire process, and my wife, who was patient with all the time I spent working on this project.

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Chapter 1. Introduction and Statement of Results

1.1 Introduction

Begin by considering the arithmetic function

$$r(n) = \#\left\{ (m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 = n \right\},$$

which is the number of ways to write an integer $n \geq 0$ as the sum of two squares. The average size of $r(n)$ can be studied analytically through the use of the summatory function

$$N(x) = \sum_{n \leq x} r(n),$$

where x is a large real number.

Geometrically, $N(x)$ can be thought as the size of the set $\{ (m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 \leq x \}$, which is the $\mathbb{Z}^2$ lattice points inside a circle of radius $\sqrt{x}$ centered at the origin. In this sense one would intuitively guess that $N(x)$ is about πx.

Define $P(x) = N(x) - \pi x$. C.F. Gauss showed using simple observations that

$$|P(x)| < \pi(\sqrt{x} + \frac{1}{2}), \tag{1.1}$$

or in short hand $P(x) = O(\sqrt{x})$. The Gauss circle problem in a general sense is the attempt to improve this estimation of $P(x)$.

In 1915 G.H. Hardy[4] and E. Landau[11] proved independently that

$$P(x) = \Omega_{-}(x^{1/4}(\log x)^{1/4}). \tag{1.2}$$

Hardy's work also included the identity

$$P(x) = \frac{x^{\frac{1}{4}}}{\pi} \sum_{n=1}^{\infty} r(n) n^{-\frac{3}{4}} \sin\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right) + \texttt{remainder terms}.$$

This, together with (1.2) suggest the conjecture that $P(x) = O(x^{1/4+\epsilon})$ for arbitrary $\epsilon > 0$.

A good survey of many of the results on this problem to date can be found in an article by A. Ivić *et. al.* [9]. There the authors mention the result of W. Seirpínski [15] that

$$P(x) = O(x^{1/3}), \tag{1.3}$$

and the the strongest result to date,

$$P(x) = O(x^{131/416}(\log x)^{18367/8320}) \tag{1.4}$$

(Huxley [6]).

The conjecture $P(x) = O(x^{1/4+\epsilon})$ is given more weight in light of the following mean square estimate,

$$\int_0^X |P(x)|^2 \, dx = \beta X^{3/2} + O(X(\log X)^2), \tag{1.5}$$

due to Kátai [10]. This result and refinements are also discussed in [9].

A natural extension of the circle problem is to replace the circle of radius $\sqrt{x}$ with a convex domain $\mathcal{D} \subset \mathbb{R}^2$. Looking at the number of lattice points in $\mathcal{D}$ gives the natural definition

$$N_{\mathcal{D}}(x) = \#\{\sqrt{x}\mathcal{D} \bigcap \mathbb{Z}^2\}, \tag{1.6}$$

where $\sqrt{x}\mathcal{D} = \left\{ (u,v) \in \mathbb{R}^2 : \sqrt{x}^{-1}(u,v) \in \mathcal{D} \right\}$ is the "blow-up" of $\mathcal{D}$ by $\sqrt{x}$.

The intuitive guess for the size of $N_{\mathcal{D}}(x)$ is $m(\mathcal{D})x$, where $m(\mathcal{D})$ denotes the area of $\mathcal{D}$.

Estimates similar to (1.2) and (1.5) for the error term,

$$P_{\mathcal{D}}(x) = N_{\mathcal{D}}(x) - m(\mathcal{D})x, \tag{1.7}$$

are given in [12] and [13] respectively. Note that these papers require the boundary of $\mathcal{D}$ to satisfy certain smoothness conditions. For discussion of domains $\mathcal{D}$ where these smoothness conditions are relaxed, consider [14], [8], and [9] for a brief survey.

Another extension of the Gauss circle problem is to count only the "primitive" lattice points inside $\mathcal{D}$ (which may be the unit disk). Primitive lattice points are those in the set $\{\, (m_1, m_2) \in \mathbb{Z}^2 : \gcd(m_1, m_2) = 1 \,\}$. These would correspond, in the case of the unit circle, to relatively prime solutions of the diophantine inequality $m_1^2 + m_2^2 \leq x$. Here the expected number of solutions is $\frac{6}{\pi^2}(\pi x)$, where a factor $\frac{6}{\pi^2}$ comes from the requirement that the solutions be relatively prime.

In the case of more general $\mathcal{D}$, write $A_{\mathcal{D}}(x)$ for the number of primitive lattice points inside $\sqrt{x}\mathcal{D}$ and let $R_{\mathcal{D}}(x) = A_{\mathcal{D}}(x) - \frac{6}{\pi^2}m(\mathcal{D})x$. Estimates for $R_{\mathcal{D}}(x)$ tend to be more difficult than those for $P_{\mathcal{D}}(x)$, and most involve the assumption of the Riemann Hypothesis (hereafter referred to as RH).

The best known bound without RH, can be found in [7, (1.6)]. This bound is

$$R_D(x) = O(x^{1/2}\omega(x)).$$

Here $\omega(x) = \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})$ with $c > 0$. In the same paper, Huxley and Nowak show that under the RH, $R_D(x) = O(x^{5/12+\epsilon})$ for arbitrary fixed $\epsilon > 0$ and sufficient smoothness conditions on the boundary of $\mathcal{D}$.

Under the RH, R. Baker [2] gives a stronger bound of

$$R_D(x) = O(x^{5/13+\epsilon})$$

with somewhat relaxed smoothness conditions on the boundary of $\mathcal{D}$. In the case that $\mathcal{D}$ is the unit disc, the exponent may be improved to $\frac{221}{608} + \epsilon$ (Wu [17]). These are the best known bounds to date.

With all the estimates that have been done on the problem of counting primitive lattice points in this context it appears that finding a mean square estimate similar to (1.5) or the result in [13] have yet to be considered. This paper is directed to this purpose.

1.2 Statement of results

We begin in the same vein as [7, proof of Lemma 1]. Let $\mathcal{D}$ be a compact convex subset of the $\mathbb{R}^2$ containing the origin. Let the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ be a C^∞ image of the unit circle and have everywhere finite nonzero curvature.

Define for $\mathbf{u} \in \mathbb{R}^2$

$$F(\mathbf{u}) = \inf\{\tau > 0 | \frac{\mathbf{u}}{\tau} \in \mathcal{D}\}$$

and $Q(\mathbf{u}) = F(\mathbf{u})^2$. Then for $m \in \mathbb{R}$, $Q(m\mathbf{u}) = m^2 Q(\mathbf{u})$.

Also define

$$N_{\mathcal{D}}(x) = \#\{\sqrt{x}\mathcal{D} \bigcap \mathbb{Z}_*^2\} = \#\{\mathbf{m} \in \mathbb{Z}_*^2 | Q(\mathbf{m}) \leq x\}$$

and

$$A_{\mathcal{D}}(x) = \#\{\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_*^2 | Q(\mathbf{m}) \leq x, \gcd(m_1, m_2) = 1\}$$

where $\mathbb{Z}_*^2 = \mathbb{Z}^2 - \{(0,0)\}$. Then

$$\begin{aligned} A_{\mathcal{D}}(x) &= \sum_{\substack{Q((m_1,m_2))\leq x \\ \gcd(m_1,m_2)=1}} 1 \\ &= \sum_{Q((m_1,m_2))\leq x} \sum_{d|(m_1,m_2)} \mu(d) \\ &= \sum_{d>0} \mu(d) N_{\mathcal{D}}(\frac{x}{d^2}) \end{aligned} \tag{1.8}$$

where $d|(m_1, m_2)$ means $d|\gcd(m_1, m_2)$ and $\mu(n)$ is the Möbius μ function.

Also let

$$Z_{\mathcal{D}}(s) = \sum_{\mathbf{m}\in\mathbb{Z}_*^2} Q(\mathbf{m})^{-s}.$$

Then we know from [7] that $Z_{\mathcal{D}}(s)$ has a meromorphic continuation on the half plane $\Re(s) > \frac{1}{4}$, with a single pole at $s = 1$ with residue $m(\mathcal{D})$.

To see this consider [7, eqn. 3.5] or Baker [2, eqn. 3.6]. It can be seen that for $\Re(s) > 1$ and $Y > 0$,

$$\begin{aligned} Z_{\mathcal{D}}(s) &= \sum_{Q(\mathbf{m})\leq Y} Q(\mathbf{m})^{-s} + \int_Y^\infty \frac{dN_{\mathcal{D}}(\omega)}{\omega^s} \\ &= \sum_{Q(\mathbf{m})\leq Y} Q(\mathbf{m})^{-s} + m(\mathcal{D}) \int_Y^\infty \omega^{-s}\, d\omega + \int_Y^\infty \frac{dP_{\mathcal{D}}(\omega)}{\omega^{-s}} \\ &= \sum_{Q(\mathbf{m})\leq Y} Q(\mathbf{m})^{-s} + \frac{m(\mathcal{D})Y^{1-s}}{s-1} - \frac{P_{\mathcal{D}}(Y)}{Y^s} + s\int_Y^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega, \end{aligned} \tag{1.9}$$

where $P_{\mathcal{D}}(x) = N_{\mathcal{D}}(x) - m(\mathcal{D})x$ is the lattice discrepancy.

Using the result that

$$\int_0^M |P_{\mathcal{D}}(\omega)|^2\, d\omega \ll (M)^{\frac{3}{2}} \tag{1.10}$$

([13], also seen in [7, eqn. 1.3]), equation (1.9) provides a meromorphic continuation of $Z_{\mathcal{D}}(s)$ to the half-plane $\Re(s) > \frac{1}{4}$, with only the simple pole of residue $m(\mathcal{D})$ at $s = 1$.

Further define,

$$f(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)}$$

where $\zeta(s)$ is the Riemann zeta function. Let $\{\lambda_k\}_{k=1}^{\infty}$ represent the increasing sequence of nonzero values taken by $Q(\mathbf{m})$ as $\mathbf{m}$ varies over $\mathbb{Z}_*^2$. Then

$$\begin{aligned} Z_{\mathcal{D}}(s) f(2s) &= \sum_{\substack{\mathbf{m}\in\mathbb{Z}_*^2 \\ n\geq 1}} \mu(n) n^{-2s} Q(\mathbf{m})^{-s} \\ &= \sum_{\substack{\mathbf{m}\in\mathbb{Z}_*^2 \\ n\geq 1}} \mu(n) Q(n\mathbf{m})^{-s} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}_*^2} \gamma(\mathbf{k}) Q(\mathbf{k}) \end{aligned}$$

where $\gamma(\mathbf{k}) = \sum_{n|(k_1,k_2)} \mu(n)$. If we define $\alpha_k = \sum_{\mathbf{n}:Q(\mathbf{n})=\lambda_k} \gamma(\mathbf{n})$ then the above equality becomes:

$$Z_{\mathcal{D}}(s) f(2s) = \sum_{k=1}^{\infty} \alpha_k \lambda_k^{-s}. \tag{1.11}$$

This prepares us for the first lemma.

Lemma 1.1. *Let $\mathcal{D}$ and $A_{\mathcal{D}}(x)$ be as above. Then we have:*

$$A_{\mathcal{D}}(x) = \frac{1}{2\pi i} \int_{3-iT}^{3+iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s}\, ds + E_1(\delta, T) + E_2(\delta)$$

where δ, T are chosen constants and

$$E_1(\delta, T) = \sum_{|x-\lambda_k|\geq\delta} O\left(\frac{x^3 \alpha_k}{\lambda_k^3 T |\log x - \log \lambda_k|} \right) \tag{1.12}$$

$$E_2(\delta) = \sum_{|x-\lambda_k|<\delta} O(|\alpha_k|) \tag{1.13}$$

Proof. Returning to (1.8)

$$A_{\mathcal{D}}(x) = \sum_{n>0} \mu(n) N_{\mathcal{D}}(\frac{x}{n^2}).$$

Rewriting this sum gives:

$$\begin{aligned} A_{\mathcal{D}}(x) &= \sum_{m>y} \mu(m) \left(\sum_{m^2 Q(\mathbf{n}) \leq x} 1 \right) \\ &= \sum_{m,\mathbf{n}:Q(m\mathbf{n})\leq x} \mu(m) \\ &= \sum_{k:\lambda_k \leq x} \alpha_k. \end{aligned}$$

Returning to (1.11) and noting the absolute convergence of the sum for $\Re(s) > 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{3-iT}^{3+iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s}\, ds &= \frac{1}{2\pi i} \int_{3-iT}^{3+iT} \sum_{k=1}^{\infty} \alpha_k \lambda_k^{-s} \frac{x^s}{s}\, ds \\ &= \sum_{k=1}^{\infty} \alpha_k \frac{1}{2\pi i} \int_{3-iT}^{3+iT} \left(\frac{x}{\lambda_k} \right)^s \frac{ds}{s}. \end{aligned}$$

Now

$$\frac{1}{2\pi i} \int_{3-iT}^{3+iT} \frac{a^s}{s}\, ds = \begin{cases} \chi(a) + O\left(\dfrac{a^3}{T|\log(a)|} \right) & (*) \\ O(a^3) & (**) \end{cases}$$

where χ is the indicator function of the interval $(1, \infty)$. Of these formulas, (*) comes from [1, p. 243] and (**) comes from [7].

Combining this formula with the integral before it gives

$$\frac{1}{2\pi i}\int_{3-iT}^{3+iT} Z_{\mathcal{D}}(s)f(2s)\frac{x^s}{s}\,ds = A_{\mathcal{D}}(x) + \sum_{|x-\lambda_k|\geq\delta} O\left(\frac{x^3\alpha_k}{\lambda_k^3 T|\log x - \log\lambda_k|}\right) + \sum_{|x-\lambda_k|<\delta} O(|\alpha_k|)$$

$$= A_{\mathcal{D}}(x) + E_1(\delta,T) + E_2(\delta)$$

where the error sum $E_1(\delta,T)$ comes from (*) and $E_2(\delta)$ comes from (**), noting that $\left(\frac{x}{\lambda_k}\right)^3 = O(1)$ when $|x-\lambda_k|<\delta$. The lemma follows at once. □

Through the use of Cauchy's integral formula rewrite the result of Lemma 1.1 as

$$\begin{aligned}\left|A_{\mathcal{D}}(x) - \frac{6}{\pi^2}m(\mathcal{D})x\right| \leq \Bigg| \frac{1}{2\pi i}\Bigg(&\int_{\frac14+\epsilon-iT}^{\frac14+\epsilon+iT} Z_{\mathcal{D}}(s)f(2s)\frac{x^s}{s}\,ds \\ &+\int_{\frac14+\epsilon-iT}^{3-iT} Z_{\mathcal{D}}(s)f(2s)\frac{x^s}{s}\,ds \\ &+\int_{\frac14+\epsilon+iT}^{3+iT} Z_{\mathcal{D}}(s)f(2s)\frac{x^s}{s}\,ds\Bigg) \\ &+E_1(\delta,T)+E_2(\delta)\Bigg|.\end{aligned} \tag{1.14}$$

Here the line of integration moves to the vertical segment along the line $\Re(s) = \frac{1}{4}+\epsilon$ for some arbitrary fixed $\epsilon > 0$. The horizontal segments and the residue at the pole are included. With appropriate estimates for the terms on the right hand side of (1.14), we shall obtain

Theorem 1.2. *Under the RH and the hypotheses on $\mathcal{D}$ as above*

$$\int_1^X \left|A_{\mathcal{D}}(x) - \frac{6}{\pi^2}m(\mathcal{D})x\right|^2 dx = O(X^{\frac{3}{2}+\epsilon})$$

for large X and any fixed $\epsilon > 0$.

Note that the necessity of the RH in this theorem comes from the estimation of the $f(2s)$ term in the integral of Lemma 1.1. It remains to prove the appropriate estimates.

2.1 Estimates for $E_1(\delta, T)$ and $E_2(\delta)$

To begin consider the error term $E_2(\delta)$.

Lemma 2.1. *Given X sufficiently large, and for $\delta \ll X^{-2}$:*

$$\int_1^X |E_2(\delta)|^2 \, dx = O(1).$$

Proof. By definition

$$\begin{aligned}\int_1^X |E_2(\delta)|^2 \, dx &= \int_1^X \left| \sum_{|x-\lambda_k|<\delta} O(|\alpha_k|) \right|^2 dx \\ &= \int_1^X \sum_{|x-\lambda_k|<\delta} \sum_{|x-\lambda_j|<\delta} O(|\alpha_k \alpha_j|) \, dx.\end{aligned} \tag{2.1}$$

Let N be the greatest integer such that $\lambda_N - \delta < X$. As

$$N \le \#\{\mathbf{m} \in \mathbb{Z}_*^2 : Q(\mathbf{m}) \le X + \delta\}$$

it is straightforward to see that $N = O(X)$ for $\delta < X$. Let $\chi_k(x)$ be the indicator function for the interval $(\lambda_k - \delta, \lambda_k + \delta)$. Then we can write the right hand side of (2.1) as

$$\int_1^X \sum_{k=1}^N \sum_{j=1}^N O(|\alpha_k \alpha_j|) \chi_k(x) \chi_j(x) \, dx. \tag{2.2}$$

Since this sum is finite, we can swap summation and integration to rewrite (2.2) as

$$\begin{aligned}\sum_{k=1}^{N}\sum_{j=1}^{N} O(|\alpha_k\alpha_j|)\int_1^X \chi_k(x)\chi_j(x)\,dx &\le \sum_{k=1}^{N}\sum_{j=1}^{N} O(|\alpha_k\alpha_j|)\int_1^X \chi_k(x)\,dx \\ &= \delta\left(\sum_{k=1}^{N} O(|\alpha_k|)\right)^2\end{aligned} \tag{2.3}$$

since $\chi_k(x)\chi_j(x) \le \chi_k(x)$.

Then as $\sum_{k=1}^{N} O(|\alpha_k|) \le N_{\mathcal{D}}(x) = O(X)$, the hypothesis $\delta = X^{-2}$ gives the lemma. □

Given this value for δ it is now possible to make an estimate for the error term $E_1(\delta, T)$. By playing a balancing act with the T term, the effect of choosing a small δ can be canceled out.

Lemma 2.2. *Given X sufficiently large and $\delta = X^{-2}$, then for $T \gg X^7$:*

$$\int_1^X |E_1(\delta, T)|^2\,dx = O(1)$$

Proof. To start, consider the term $|\log x - \log\lambda_k|^{-1}$. By the mean value theorem for positive x, λ_k:

$$|\log x - \log\lambda_k|^{-1} \ll \frac{\max(x, \lambda_k)}{|x-\lambda_k|} \ll \frac{x\lambda_k}{|x-\lambda_k|}.$$

So that

$$\begin{aligned}E_1(\delta, T) &= \sum_{|x-\lambda_k|\ge\delta} O\left(\frac{x^3\alpha_k}{\lambda_k^3 T|\log x - \log\lambda_k|}\right) \\ &\le \sum_{|x-\lambda_k|\ge\delta} O\left(\frac{x^4\alpha_k}{\lambda_k^2 T|x-\lambda_k|}\right) \\ &\le \sum_{|x-\lambda_k|\ge\delta} O\left(\frac{x^4\alpha_k}{\lambda_k^2 T\delta}\right).\end{aligned} \tag{2.4}$$

This gives

$$\begin{aligned}\int_1^X |E_1(\delta,T)|^2 \, dx &= \int_1^X \left| \sum_{|x-\lambda_k|\geq\delta} O\left(\frac{x^4\alpha_k}{\lambda_k^2 T\delta}\right)\right|^2 \, dx \\ &= \int_1^X \left| \frac{x^4}{T\delta} \sum_{|x-\lambda_k|\geq\delta} O\left(\frac{\alpha_k}{\lambda_k^2}\right)\right|^2 \, dx \\ &\leq \frac{1}{(T\delta)^2} \int_1^X x^8 \left| \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^2}\right|^2 \, dx.\end{aligned} \tag{2.5}$$

Then as $\sum_{k=1}^{\infty} \alpha_k \lambda_k^{-s}$ is absolutely convergent for $\Re(s) > 1$, (2.5) shows

$$\int_1^X |E_1(\delta,T)|^2 \, dx = O\left(\frac{X^9}{T^2\delta^2}\right).$$

The hypothesis $T \gg X^7$ gives the lemma. □

2.2 An Estimate for the Horizontal Segments

Begin with an important lemma

Lemma 2.3. *Under the RH for* $\sigma > \frac{1}{2}$, $|t| \geq 1$, *and* $0 < y < 1$

$$|f(\sigma + it)| \ll |t|^{\epsilon}$$

for every $\epsilon > 0$.

Proof. This is a fairly well known lemma that is proved in all essentials in [16, §14.25]. □

With this lemma it is possible to consider the integrals of the inequality (1.14). Restricting for the moment to the horizontal components, the first priority is to get a good understanding of the integration along the contour. This is so because a bound on

$$\int_1^X \left| \int_{\frac{1}{4}+\epsilon \pm iT}^{3\pm iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s} \, ds \right|^2 \, dx \tag{2.6}$$

is effectively given by a bound on

$$\int_{\frac{1}{4}+\epsilon}^{3} \left| Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s} \right| \, d\sigma \tag{2.7}$$

for $s = \sigma + it$, $T \le t \le 2T$. Note that both the upper horizontal contour and the lower horizontal contour can be treated the same by symmetry because the integrands on the horizontal segments are complex conjugates of each other.

With the strength of Lemma 2.3 and some straightforward pointwise bounds it is possible to get that equation (2.7) is

$$\ll x^3 T^\epsilon \int_{\frac{1}{4}+\epsilon}^{3} \frac{|Z_{\mathcal{D}}(\sigma + it)|}{|\sigma + it|} \, d\sigma,$$

but to get any further the following lemmas are required.

Lemma 2.4. *Let $A > 0$, $A < B \le 2A$, $C \ge 2$, $C < D \le 2C$. Let f be a bounded measurable function on $[A, B]$. Then*

$$\int_C^D \left| \int_A^B h(x) x^{it} \, dx \right|^2 \ll A \log C \int_A^B |h(x)|^2 \, dx.$$

Proof. This is [3, Lemma 5]. □

Lemma 2.5. *For $T \ge 1$, $\sigma \ge \frac{1}{4} + \epsilon$,*

$$M(Z_{\mathcal{D}}) = \int_T^{2T} |Z_{\mathcal{D}}(\sigma + it)|^2 \, dt \ll T^2 \log T.$$

Proof. This lemma is a variant of [3, Lemma 6].

To start take $Y = 1$ in (1.9) to get

$$Z_{\mathcal{D}}(s) = s\int_1^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega + O(1) \tag{2.8}$$

for s not too close to 1, $|s-1| \geq \frac{1}{100}$ say.

Then for $s = \sigma + it$

$$\begin{aligned} M(Z_{\mathcal{D}}) &= \int_T^{2T} \left| s\int_1^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega + O(1)\right|^2\, dt \\ &\ll T + T^2\int_T^{2T}\left|\int_1^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega\right|^2\, dt \\ &\ll T + T^2\int_T^{2T}\left|\sum_{j=1}^\infty \int_{2^{j-1}}^{2^j} \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega\right|^2\, dt. \end{aligned} \tag{2.9}$$

Applying Cauchy's inequality yields

$$M(Z_{\mathcal{D}}) \ll T + T^2\int_T^{2T}\left(\sum_{j=1}^\infty j^{-2}\right)\left(\sum_{j=1}^\infty j^2\left|\int_{2^{j-1}}^{2^j}\frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\, d\omega\right|^2\right).\, dt \tag{2.10}$$

Applying (1.10) gives

$$\int_0^{2^j} |P_{\mathcal{D}}(\omega)|^2\, d\omega \ll (2^j)^{\frac{3}{2}}.$$

So for $\sigma \geq \frac{1}{4} + \epsilon$

$$\begin{aligned} \int_{2^{j-1}}^{2^j} \frac{|P_{\mathcal{D}}(\omega)|^2}{\omega^{2\sigma+2}}\, d\omega &\ll \int_{2^{j-1}}^{2^j}\frac{|P_{\mathcal{D}}(\omega)|^2}{\omega^{\frac{5}{2}+\epsilon}}\, d\omega \\ &\ll 2^{-j(\frac{5}{2}+\epsilon)}\int_{2^{j-1}}^{2^j}|P_{\mathcal{D}}(\omega)|^2\, d\omega \\ &\ll 2^{-j(\frac{5}{2}+\epsilon)}\int_0^{2^j}|P_{\mathcal{D}}(\omega)|^2\, d\omega \\ &\ll 2^{-j(1+\epsilon)} \end{aligned} \tag{2.11}$$

as $P_{\mathcal{D}}(\omega)$ is an increasing function of ω.

Applying lemma 2.4 with $f(\omega) = P_{\mathcal{D}}(\omega)\omega^{-s-1}$ gives

$$\int_T^{2T} \left| \int_{2^{j-1}}^{2^j} \frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}} \, d\omega \right|^2 \, dt \ll 2^j \log T (2^{-j(1+\epsilon)}) = 2^{-j\epsilon} \log T$$

so that

$$M(Z_{\mathcal{D}}) \ll T + T^2 \left(\sum_{j=1}^{\infty} j^2 2^{-j\epsilon} \right) \log T \ll T^2 \log T$$

□

Lemma 2.6. *If* $T \geq 1$, $\sigma \geq \frac{1}{4} + \epsilon$ *and*

$$\int_T^{2T} |Z_{\mathcal{D}}(\sigma + it)|^2 \, dt \ll T^{2+\epsilon}$$

for any fixed $\epsilon > 0$, *then there exists some* t_0 *satisfying* $T \leq t_0 \leq 2T$ *such that*

$$\int_{\frac{1}{4}+\epsilon}^{3} \frac{|Z_{\mathcal{D}}(\sigma + it_0)|}{|\sigma + it_0|} \, d\sigma \ll T^{-1/2+\epsilon}.$$

Proof. By Cauchy's inequality, for $\sigma \geq \frac{1}{4} + \epsilon$,

$$\int_T^{2T} |Z_{\mathcal{D}}(\sigma + it)| \, dt \leq \left(\int_T^{2T} 1 \, dt \right)^{\frac{1}{2}} \left(\int_T^{2T} |Z_{\mathcal{D}}(\sigma + it)|^2 \, dt \right)^{\frac{1}{2}}$$
$$\ll T^{\frac{3}{2}+\epsilon}.$$

Hence

$$\int_T^{2T} \left\{ \int_{\frac{1}{4}+\epsilon}^{3} \frac{|Z_{\mathcal{D}}(\sigma + it)|}{|\sigma + it|} \, d\sigma \right\} \, dt \ll T^{\frac{1}{2}+\epsilon}.$$

The Lemma follows at once. □

Lemma 2.5 implies Lemma 2.6 and the bound on (2.7) improves to

$$\ll x^3 T^\epsilon \int_{\frac{1}{4}+\epsilon}^{3} \frac{|Z_{\mathcal{D}}(\sigma+it)|}{|\sigma+it|}\, d\sigma \ll \frac{x^3}{T^{\frac{1}{2}-2\epsilon}}.$$

This together with the fact that $T \gg X^7$ gives:

Lemma 2.7. *Given X sufficiently large, the expression in (2.6) is $O(1)$ for $T \gg X^7$*

2.3 An Estimate for the Left-Hand contour

Begin first with the following lemma.

Lemma 2.8. *Let $D > C \geq 2$, $B > A > 1$ and suppose that $h(t)$ is a bounded measurable function on $[C, D]$. Then*

$$\int_A^B \left| \int_C^D h(t) x^{it}\, dt \right|^2 \ll B \log D \int_C^D |h(t)|^2\, dt.$$

Proof. This is a slight variant of [5, Lemma 9.1] or [3, Lemma 7]. □

Now let $\lambda = \frac{1}{4} + \epsilon$, $g(t) = \frac{Z_{\mathcal{D}}(\lambda+it) f(2(\lambda+it))}{\lambda+it}$. Then the contribution of the left-hand contour integral is given by:

Lemma 2.9. *Given X sufficiently large and some arbitrary $\epsilon > 0$,*

$$\int_1^X \left| \int_{-T}^T g(t) x^{\lambda+it}\, dt \right|^2 dx = O(X^{\frac{3}{2}+\epsilon}). \tag{2.12}$$

Proof. Beginning with the obvious estimate

$$\int_1^X \left| \int_{-T}^T g(t) x^{\lambda+it}\, dt \right|^2 dx \leq \int_1^X x^{2\lambda} \left| \int_{-T}^T g(t) x^{it}\, dt \right|^2 dx$$

it can be seen that with a bit of work a version of Lemma 2.8 may be applied.

Now

$$\left|\int_{-T}^{T} g(t)x^{it}\,dt\right|^2 = \left|\int_{-2}^{2} g(t)x^{it}\,dt + \int_{2}^{T} g(t)x^{it} + g(-t)x^{-it}\,dt\right|^2$$
$$\ll 1 + O\left(\left|\int_{2}^{T} g(t)x^{it}\,dt\right|^2\right).$$

So letting $N = 2^{-k}T \ \ (k = 1, 2, \dots)$ so that $2 \le N \le T$ write

$$\begin{aligned}\left|\int_{2}^{T} g(t)x^{it}\,dt\right|^2 &= \left|\sum_N \int_N^{2N} g(t)x^{it}\,dt\right|^2 + O(1) \\ &\ll (\log T)^2 \left|\int_U^{2U} g(t)x^{it}\,dt\right|^2 + 1\end{aligned} \tag{2.13}$$

for some U, $2 \le U \le T$ such that the integral inside the sum is maximized. Then

$$\left|\int_{-T}^{T} g(t)x^{it}\,dt\right|^2 \ll 1 + (\log T)^2 \left|\int_U^{2U} g(t)x^{it}\,dt\right|^2$$

which gives

$$\begin{aligned}\int_1^X x^{2\lambda}\left|\int_{-T}^{T} g(t)x^{it}\,dt\right|^2 dx &\ll \int_1^X x^{2\lambda}\,dx + (\log T)^2 \int_1^X x^{2\lambda}\left|\int_U^{2U} g(t)x^{it}\,dt\right|^2 dx \\ &\le X^{2\lambda}\left[X + (\log T)^2 \int_1^X \left|\int_U^{2U} g(t)x^{it}\,dt\right|^2 dx\right] \\ &\le X^{2\lambda+1} + X^{2\lambda}(\log T)^2 \int_1^X \left|\int_U^{2U} g(t)x^{it}\,dt\right|^2 dx\end{aligned} \tag{2.14}$$

as

$$\int_1^X x^{2\lambda} f(x)\,dx \le X^{2\lambda}\int_1^X f(x)\,dx$$

for $X > 1$.

Then applying Lemma 2.8 to the integral on the right-hand side of (2.14) with $h(t) = g(t)$

gives

$$\int_1^X \left| \int_U^{2U} g(t) x^{it} \, dt \right|^2 \, dx \ll X \log U \int_U^{2U} |g(t)|^2 \, dt. \tag{2.15}$$

Using pointwise estimates and Lemma 2.3 gives a bound

$$\ll \frac{XU^\epsilon \log U}{U^2} \int_U^{2U} |Z_{\mathcal{D}}(\lambda + it)|^2 \, dt.$$

By Lemma 2.5 this is

$$\ll XU^\epsilon (\log U)^2 \ll XT^\epsilon (\log T)^2$$

Since $U \leq T$.

Combining the above bound with (2.14) gives

$$\int_1^X \left| \int_{-T}^T g(t) x^{\lambda + it} \, dt \right|^2 \, dx = O(X^{\frac{3}{2}+2\epsilon}) + O(X^{\frac{3}{2}+2\epsilon} T^\epsilon (\log T)^4)$$

and since $\epsilon > 0$ was arbitrary, this gives the result. □

As mentioned before, Theorem 1.2 is proved by combining the inequality (1.14) with Lemmas 2.1, 2.2, 2.7, and 2.9. Some care is required in saying this though.

Applying mean square integration to (1.14) directly gives

$$\begin{aligned} \int_1^X \left| A_{\mathcal{D}}(x) - \frac{6}{\pi^2} m(\mathcal{D}) x \right|^2 dx \leq \int_1^X \Bigg| \frac{1}{2\pi i} \Bigg(& \int_{\frac{1}{4}+\epsilon-iT}^{\frac{1}{4}+\epsilon+iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s} \, ds \\ & + \int_{\frac{1}{4}+\epsilon-iT}^{3-iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s} \, ds \\ & + \int_{\frac{1}{4}+\epsilon+iT}^{3+iT} Z_{\mathcal{D}}(s) f(2s) \frac{x^s}{s} \, ds \Bigg) \\ & + E_1(\delta, T) + E_2(\delta) \Bigg|^2 dx, \end{aligned} \tag{2.16}$$

and the right hand side of (2.16) does not split up directly into the pieces discussed by

Lemmas 2.1, 2.2, 2.7, and 2.9. However, from Cauchy-Schwarz inequality,

$$\int_1^X \left|\sum_{j=1}^{n} F_j(x)\right|^2 \, dx \leq n \sum_{j=1}^{n} \int_1^X |F_j(x)|^2 \, dx.$$

Taking $n = 5$ in the above expression and letting the $F_j(x)$ run through the terms on the right hand side of (2.16) gives Theorem 1.2.

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